J. Math. Anal. Appl. 375 (2011) 230–244

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Fully discrete finite element scheme for Maxwell's equations with non-linear boundary condition

Marián Slodička *, Stephane Durand

Research Group NaM[2], Department of Mathematical Analysis, Ghent University, Galglaan 2, 9000 Gent, Belgium

ARTICLE INFO

Article history:
Received 8 July 2010
Available online 16 September 2010
Submitted by Goong Chen

Keywords:
Maxwell's equations
Non-linear Silver–Müller boundary condition
Time discretization
Convergence
Conforming finite elements
Error estimates

ABSTRACT

We study a full Maxwell's system accompanied with a non-linear degenerate boundary condition, which represents a generalization of the classical Silver–Müller condition for a non-perfect conductor. The relationship between the normal components of electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ field obeys the following power law $\boldsymbol{\nu} \times \boldsymbol{H} = \boldsymbol{\nu} \times (|\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu})$ for some $\alpha \in (0, 1]$. We establish the existence and uniqueness of a weak solution in a suitable function spaces under the minimal regularity assumptions on the boundary Γ and the initial data $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. We design a non-linear time discrete approximation scheme and prove convergence of the approximations to a weak solution. We also derive the error estimates for the time discretization. As a next step we study the fully discrete problem using curl-conforming edge elements and derive the corresponding error estimates. Finally we present some numerical experiments.

1. Introduction

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^3$ (with a Lipschitz boundary Γ), which is occupied by a ferromagnetic material. The electromagnetic field in Ω can be described by several vector fields, i.e., $\boldsymbol{B}$ – magnetic induction, $\boldsymbol{H}$ – magnetic field, $\boldsymbol{E}$ – electric field, $\boldsymbol{D}$ – electric displacement field, and $\boldsymbol{J}$ – free current density.

We consider the Maxwell equations of the form

$$\begin{aligned} &\partial_t \boldsymbol{D} - \nabla \times \boldsymbol{H} + \boldsymbol{J} = \boldsymbol{0}, \quad \text{Ampère's law}, \\ &\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = \boldsymbol{0}, \quad \text{Faraday's law}, \\ &\boldsymbol{J} = \boldsymbol{J}_0 + \sigma \boldsymbol{E}, \quad \text{Ohm's law}, \end{aligned} \tag{1}$$

where $\boldsymbol{J}_0$ is a given vector field and σ denotes the conductivity.

We assume linear magnetic materials, i.e.,

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}, \tag{2}$$

where μ denotes the magnetic permeability and ε is the permittivity of the corresponding materials.

Eliminating the quantities $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{J}$ in (1)–(2) we get

$$\begin{aligned} &\varepsilon \partial_t \boldsymbol{E} + \sigma \boldsymbol{E} - \nabla \times \boldsymbol{H} = -\boldsymbol{J}_0, \\ &\mu \partial_t \boldsymbol{H} + \nabla \times \boldsymbol{E} = \boldsymbol{0}. \end{aligned} \tag{3}$$

* Corresponding author.
E-mail address: ms@cage.ugent.be (M. Slodička).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.09.016

This system (3) will be accompanied with a non-linear boundary condition (BC) between the normal components of $\boldsymbol{H}$ and $\boldsymbol{E}$, which represents a non-perfect contact of different materials at the boundary Γ [1–3]. In this paper we consider a power law non-linearity of the form

$$\boldsymbol{H} \times \boldsymbol{\nu} = \boldsymbol{\nu} \times \boldsymbol{g}(\boldsymbol{E} \times \boldsymbol{\nu}) = \boldsymbol{\nu} \times \big(|\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu}\big), \quad \alpha \in (0,1]. \tag{4}$$

We can also consider a more general function $\boldsymbol{g}$, but then we have to adopt some assumptions ensuring the monotonicity, hemi-continuity and coercivity of the non-linear operator in appropriate function spaces.

In the case when $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}$, the BC (4) represents the classical Silver–Müller condition, which (cf. [4,5]) is a first order approximation of the so-called "transparent" boundary condition. Sometimes it is also called Leontovich or impedance BC, cf. [6,7].

The decay rates for the energy for the full Maxwell system have been derived in [1]. The Galerkin approximation of a solution for a linear Silver–Müller BC has been studied in [8]. Quasi-static Maxwell's equations with the non-linear boundary condition (4) have been studied in [9,10].

The stabilization of Maxwell's equations with space–time variable coefficients by means of linear or non-linear Silver–Müller boundary condition was discussed in [11]. This is based on some stability estimates that are obtained using the standard identity with multiplier and appropriate properties of the feedback.

The main goal of this paper is to design a fully-discrete numerical scheme for the approximation of an exact solution to this Maxwell system with the non-linear boundary condition. Error estimates for the approximation of linear Maxwell's equations are studied in several papers. We refer the reader to [12–14], where optimal error estimates are proved for a conforming finite element scheme based on Nédélec's elements. In this paper, we prove sub-optimal convergence rate for a fully discrete finite element scheme to approximate a system with non-linear boundary conditions. So far, the authors could not prove optimality, due to the non-linearity on the boundary.

We start with the study of a semi-discrete approximation scheme, based on backward Euler's method in Section 2. In Section 3, we prove some stability results for semi-discrete approximations. Then, employing the theory of monotone operators, we prove convergence and derive the corresponding error estimates in Section 4 under minimal regularity assumptions on the boundary Γ and the initial data $\boldsymbol{H}_0, \boldsymbol{E}_0$. As a next step, we propose a fully discrete finite element scheme in Section 5 and derive error estimates. Finally, we support the theoretical results by some basic numerical examples.

2. Time discretization

We shall work in a variational framework. We denote by $(\boldsymbol{w}, \boldsymbol{z})$ the usual $\mathbf{L}_2$-inner product of vector-valued functions $\boldsymbol{w}$ and $\boldsymbol{z}$ in Ω, i.e., $(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{z}$ and $\|\boldsymbol{w}\| = \sqrt{(\boldsymbol{w}, \boldsymbol{w})}$. The $\mathbf{L}_2$-inner product on the boundary Γ will be written as $(\boldsymbol{w}, \boldsymbol{z})_\Gamma = \int_\Gamma \boldsymbol{w} \cdot \boldsymbol{z}$. We shall use standard function spaces $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{L}_p(\Gamma)$ for some $p > 1$, see [15]. The norm in $\mathbf{H}(\mathbf{curl}, \Omega)$ is defined as

$$\|\boldsymbol{\varphi}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} = \|\boldsymbol{\varphi}\|^2 + \|\nabla \times \boldsymbol{\varphi}\|^2.$$

The space of test functions will be denoted by

$$\boldsymbol{V}_p = \big\{\boldsymbol{\varphi} \in \mathbf{H}(\mathbf{curl}, \Omega);\ \boldsymbol{\varphi} \times \boldsymbol{\nu} \in \mathbf{L}_p(\Gamma)\big\},$$

which will be a natural choice for our problem (3) and (4). We recall that $\boldsymbol{V}_p$ is a reflexive Banach space,[1] which will be endowed with the sum-norm $\|\boldsymbol{\varphi}\|_{\boldsymbol{V}_p} = \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\varphi} \times \boldsymbol{\nu}\|_{\mathbf{L}_p(\Gamma)}$. Finally, we introduce the space

$$\mathbf{H}^1(\mathbf{curl}, \Omega) = \big\{\mathbf{u} \in \mathbf{H}^1(\Omega) \;\big|\; \nabla \times \mathbf{u} \in \mathbf{H}^1(\Omega)\big\},$$

necessary for estimation of the approximation properties of the edge element interpolation operator [12,17,15].

For ease of exposition, we set $\mu = \varepsilon = \sigma = 1$ and $\boldsymbol{J}_0 = \mathbf{0}$, in order to focus on the non-linearity in the problem setting. Let us note that our approach and all the results will also be valid for the case if $\sigma = 0$.

The variational formulation of (3), (4) together with initial conditions reads as

$$\begin{aligned}
&(\partial_t \boldsymbol{E}, \boldsymbol{\varphi}) + (\boldsymbol{E}, \boldsymbol{\varphi}) - (\boldsymbol{H}, \nabla \times \boldsymbol{\varphi}) + \big(|\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \times \boldsymbol{\nu}\big)_\Gamma = 0,\\
&(\partial_t \boldsymbol{H}, \boldsymbol{\psi}) + (\nabla \times \boldsymbol{E}, \boldsymbol{\psi}) = 0,\\
&\boldsymbol{H}(0) = \boldsymbol{H}_0,\\
&\boldsymbol{E}(0) = \boldsymbol{E}_0
\end{aligned} \tag{5}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{V}_{1+\alpha}$ and $\boldsymbol{\psi} \in \mathbf{L}_2(\Omega)$ and for almost all $t \in (0, T)$. We assume that the initial conditions are compatible with the boundary condition, i.e., $\boldsymbol{H}_0$ and $\boldsymbol{E}_0$ satisfy the non-linear boundary condition.

[1] This follows from the fact that both $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{L}_p(\Gamma)$ are reflexive Banach spaces – see [16, Theorem 5.13].

The time discretization is based on backward Euler's method. We use an equidistant partitioning with a time step $\tau = \frac{T}{n}$, for any $n \in \mathbb{N}$. Therefore, we divide the time interval $[0, T]$ into n subintervals $[t_{i-1}, t_i]$ for $t_i = i\tau$. We introduce the following notation

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

We suggest the following non-linear recurrent approximation scheme for $i = 1, \ldots, n$ and $\boldsymbol{\varphi} \in \boldsymbol{V}_{1+\alpha}$ and $\boldsymbol{\psi} \in \mathbf{L}_2(\Omega)$,

$$\begin{aligned}
&(\delta \boldsymbol{e}_i, \boldsymbol{\varphi}) + (\boldsymbol{e}_i, \boldsymbol{\varphi}) - (\boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) + \left(|\boldsymbol{e}_i \times \boldsymbol{\nu}|^{\alpha-1} \boldsymbol{e}_i \times \boldsymbol{\nu}, \boldsymbol{\varphi} \times \boldsymbol{\nu}\right)_\Gamma = 0,\\
&(\delta \boldsymbol{h}_i, \boldsymbol{\psi}) + (\nabla \times \boldsymbol{e}_i, \boldsymbol{\psi}) = 0,\\
&\boldsymbol{h}_0 = \boldsymbol{H}_0,\\
&\boldsymbol{e}_0 = \boldsymbol{E}_0.
\end{aligned} \tag{6}$$

This scheme can be rewritten as follows

$$\begin{aligned}
&\left(\frac{1+\tau}{\tau} \boldsymbol{e}_i, \boldsymbol{\varphi}\right) - (\boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) + \left(|\boldsymbol{e}_i \times \boldsymbol{\nu}|^{\alpha-1} \boldsymbol{e}_i \times \boldsymbol{\nu}, \boldsymbol{\varphi} \times \boldsymbol{\nu}\right)_\Gamma = \left(\frac{\boldsymbol{e}_{i-1}}{\tau}, \boldsymbol{\varphi}\right),\\
&\left(\frac{\boldsymbol{h}_i}{\tau}, \boldsymbol{\psi}\right) + (\nabla \times \boldsymbol{e}_i, \boldsymbol{\psi}) = \left(\frac{\boldsymbol{h}_{i-1}}{\tau}, \boldsymbol{\psi}\right)
\end{aligned}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{V}_{1+\alpha}$ and $\boldsymbol{\psi} \in \mathbf{L}_2(\Omega)$.

In order to prove existence and uniqueness of a weak solution $\boldsymbol{e}_i, \boldsymbol{h}_i$ for each time step t_i, $i = 1, \ldots, n$, we will need the following lemma [18, Lemma 18.2].

Lemma 2.1. *Let D be an open bounded set in euclidean space E, containing zero, and G a continuous mapping of the closure $\overline{D}$ into E. If*

$$\big(G(x), x\big) > 0,$$

on the boundary of D, then the equation $G(x) = 0$ has at least one solution $x_0 \in D$.

We can now prove the following lemma.

Lemma 2.2. *Assume $\boldsymbol{E}_0, \boldsymbol{H}_0 \in \mathbf{L}_2(\Omega)$. Then there exist uniquely determined fields $\boldsymbol{e}_i \in \boldsymbol{V}_{1+\alpha}$ and $\boldsymbol{h}_i \in \mathbf{L}_2(\Omega)$ solving* (6) *for any $i = 1, \ldots, n$.*

Proof. Suppose that we want to prove the existence and uniqueness of a solution for the following system

$$\begin{aligned}
&(\boldsymbol{e}, \boldsymbol{\varphi}) - (\boldsymbol{h}, \nabla \times \boldsymbol{\varphi}) + \left(|\boldsymbol{e} \times \boldsymbol{\nu}|^{\alpha-1} \boldsymbol{e} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \times \boldsymbol{\nu}\right)_\Gamma = (\boldsymbol{a}, \boldsymbol{\varphi}),\\
&(\boldsymbol{h}, \boldsymbol{\psi}) + (\nabla \times \boldsymbol{e}, \boldsymbol{\psi}) = (\boldsymbol{b}, \boldsymbol{\psi})
\end{aligned} \tag{7}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{V}_{1+\alpha}$, $\boldsymbol{\psi} \in \mathbf{L}_2(\Omega)$ and given $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{L}_2(\Omega)$.

We approximate the spaces $\boldsymbol{V}_{1+\alpha}$ and $\mathbf{L}_2(\Omega)$ by the finite dimensional subspaces $\boldsymbol{V}^k_{1+\alpha} = [\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_k]$ and $\mathbf{L}^k_2 = [\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_k]$ respectively, with $\nabla \times \boldsymbol{V}^k_{1+\alpha} \subset \mathbf{L}^k_2$. We assume the following approximation property

$$\lim_{k\to\infty} \|\boldsymbol{\varphi} - P_{\boldsymbol{V}^k_{1+\alpha}} \boldsymbol{\varphi}\|_{\boldsymbol{V}_{1+\alpha}} = \lim_{k\to\infty} \|\boldsymbol{\psi} - P_{\mathbf{L}^k_2} \boldsymbol{\psi}\|_{\mathbf{L}_2(\Omega)} = 0, \tag{8}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{V}_{1+\alpha}$, $\boldsymbol{\psi} \in \mathbf{L}_2(\Omega)$. Here $P_{\boldsymbol{V}^k_{1+\alpha}}$ denotes the projector onto $\boldsymbol{V}^k_{1+\alpha}$ and is defined such that for every $\boldsymbol{\varphi} \in \boldsymbol{V}_{1+\alpha}$,

$$\|P_{\boldsymbol{V}^k_{1+\alpha}} \boldsymbol{\varphi} - \boldsymbol{\varphi}\|_{\boldsymbol{V}_{1+\alpha}} \leqslant \left\|\boldsymbol{\varphi}^k - \boldsymbol{\varphi}\right\|_{\boldsymbol{V}_{1+\alpha}}, \quad \forall \boldsymbol{\varphi}^k \in \boldsymbol{V}^k_{1+\alpha},$$

and $P_{\mathbf{L}^k_2}$ is the standard orthogonal projector in $\mathbf{L}_2(\Omega)$.

The finite dimensional approximation of (7) reads as

$$\begin{aligned}
&(\boldsymbol{e}_k, \boldsymbol{\varphi}) - (\boldsymbol{h}_k, \nabla \times \boldsymbol{\varphi}) + \left(|\boldsymbol{e}_k \times \boldsymbol{\nu}|^{\alpha-1} \boldsymbol{e}_k \times \boldsymbol{\nu}, \boldsymbol{\varphi} \times \boldsymbol{\nu}\right)_\Gamma = (\boldsymbol{a}, \boldsymbol{\varphi}),\\
&(\boldsymbol{h}_k, \boldsymbol{\psi}) + (\nabla \times \boldsymbol{e}_k, \boldsymbol{\psi}) = (\boldsymbol{b}, \boldsymbol{\psi})
\end{aligned} \tag{9}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{V}^k_{1+\alpha}$, $\boldsymbol{\psi} \in \mathbf{L}^k_2$. The approximations have the form

$$\boldsymbol{e}_k = \sum_{i=1}^k \lambda_i \boldsymbol{\varphi}_i, \qquad \boldsymbol{h}_k = \sum_{i=1}^k \xi_i \boldsymbol{\psi}_i.$$

Let us denote $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k)$. We are going to apply the theory of monotone operators (see [16,18]) to show the existence of a weak solution to the boundary value problem (9). We can look at (9) as on a non-linear operator $\boldsymbol{A}(\boldsymbol{\lambda}, \boldsymbol{\xi}) : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k$ defined by

$$\begin{aligned}
\boldsymbol{A}^{2j-1}(\boldsymbol{\lambda}, \boldsymbol{\xi}) &:= \left(\sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j \right) - \left(\sum_{i=1}^{k} \xi_i \boldsymbol{\psi}_i, \nabla \times \boldsymbol{\varphi}_j \right) - (\boldsymbol{a}, \boldsymbol{\varphi}_j) \\
&\quad + \left(\left| \left(\sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i \right) \times \boldsymbol{\nu} \right|^{\alpha-1} \left(\sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i \right) \times \boldsymbol{\nu}, \boldsymbol{\varphi}_j \times \boldsymbol{\nu} \right)_{\Gamma}, \\
\boldsymbol{A}^{2j}(\boldsymbol{\lambda}, \boldsymbol{\xi}) &:= \left(\sum_{i=1}^{k} \xi_i \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \right) + \left(\nabla \times \sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i, \boldsymbol{\psi}_j \right) - (\boldsymbol{b}, \boldsymbol{\psi}_j)
\end{aligned}$$

for any $j = 1, \ldots, k$.

Now, we introduce the non-linear operator $\boldsymbol{a}(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$\boldsymbol{a}\big(\boldsymbol{x}\big) := g\big(|\boldsymbol{x}|\big)\boldsymbol{x} = |\boldsymbol{x}|^{\alpha-1}\boldsymbol{x}. \tag{10}$$

The gradient of $\boldsymbol{a}(\boldsymbol{x})$ in the direction $\boldsymbol{y}$ is

$$\big\langle \operatorname{grad} \boldsymbol{a}(\boldsymbol{x}), \boldsymbol{y} \big\rangle = \big\langle \operatorname{grad} g\big(|\boldsymbol{x}|\big)\boldsymbol{x}, \boldsymbol{y} \big\rangle = g'\big(|\boldsymbol{x}|\big) \frac{\boldsymbol{y} \cdot \boldsymbol{x}}{|\boldsymbol{x}|} \boldsymbol{x} + g\big(|\boldsymbol{x}|\big) \boldsymbol{y}.$$

The monotonicity of $\boldsymbol{a}(\boldsymbol{x})$ follows for some $\theta \in (0, 1)$ from

$$\begin{aligned}
\big[\boldsymbol{a}(\boldsymbol{x} + \boldsymbol{y}) - \boldsymbol{a}(\boldsymbol{x})\big] \cdot \boldsymbol{y} &= \big\langle \operatorname{grad} \boldsymbol{a}(\boldsymbol{x} + \theta \boldsymbol{y}), \boldsymbol{y} \big\rangle \cdot \boldsymbol{y} \\
&= g\big(|\boldsymbol{x} + \theta \boldsymbol{y}|\big)|\boldsymbol{y}|^2 + g'\big(|\boldsymbol{x} + \theta \boldsymbol{y}|\big) \frac{(\boldsymbol{y} \cdot (\boldsymbol{x} + \theta \boldsymbol{y}))^2}{|\boldsymbol{x} + \theta \boldsymbol{y}|} \\
&\geqslant \big[g\big(|\boldsymbol{x} + \theta \boldsymbol{y}|\big) - \big|g'\big(|\boldsymbol{x} + \theta \boldsymbol{y}|\big)\big| |\boldsymbol{x} + \theta \boldsymbol{y}| \big] |\boldsymbol{y}|^2 \\
&\geqslant \alpha |\boldsymbol{x} + \theta \boldsymbol{y}|^{\alpha-1} |\boldsymbol{y}|^2 \\
&\geqslant 0.
\end{aligned} \tag{11}$$

Further, we can write

$$\begin{aligned}
\big(\boldsymbol{A}(\boldsymbol{\lambda}, \boldsymbol{\xi}), (\boldsymbol{\lambda}, \boldsymbol{\xi})\big) &= \left\| \sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i \right\|^2 + \left\| \sum_{i=1}^{k} \xi_i \boldsymbol{\psi}_i \right\|^2 + \int_{\Gamma} \left| \left(\sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i \right) \times \boldsymbol{\nu} \right|^{1+\alpha} - \left(\boldsymbol{a}, \sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i \right) - \left(\boldsymbol{b}, \sum_{i=1}^{k} \xi_i \boldsymbol{\psi}_i \right) \\
&\geqslant \frac{1}{2} \left(\left\| \sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i \right\|^2 + \left\| \sum_{i=1}^{k} \xi_i \boldsymbol{\psi}_i \right\|^2 - \|a\|^2 - \|b\|^2 \right) + \int_{\Gamma} \left| \left(\sum_{i=1}^{k} \lambda_i \boldsymbol{\varphi}_i \right) \times \boldsymbol{\nu} \right|^{1+\alpha} \\
&\geqslant C_0\big(|\boldsymbol{\lambda}|^2 + |\boldsymbol{\xi}|^2\big) - C\big(\|a\|^2 + \|b\|^2\big)
\end{aligned} \tag{12}$$

for some positive constants C and C_0.

So, $(\boldsymbol{A}(\boldsymbol{\lambda}, \boldsymbol{\xi}), (\boldsymbol{\lambda}, \boldsymbol{\xi})) > 0$ if $|\boldsymbol{\lambda}|^2 + |\boldsymbol{\xi}|^2 = r^2$ provided we select $r > 0$ sufficiently large. We apply Lemma 2.1, to conclude that $\boldsymbol{A}(\boldsymbol{\lambda}_k, \boldsymbol{\xi}_k) = \boldsymbol{0}$ for some vector $(\boldsymbol{\lambda}_k, \boldsymbol{\xi}_k) \in \mathbb{R}^k \times \mathbb{R}^k$. This also implies the existence of $\boldsymbol{e}_k \in \boldsymbol{V}^k_{1+\alpha}, \boldsymbol{h}_k \in \mathbf{L}^k_2$ which solve (9).

Now, we derive some a priori estimates for $\boldsymbol{e}_k, \boldsymbol{h}_k$ uniform to k. We set $\boldsymbol{\varphi} = \boldsymbol{e}_k$ and $\boldsymbol{\psi} = \boldsymbol{h}_k$ in (9) and we sum up both equations. We get

$$\|\boldsymbol{e}_k\|^2 + \|\boldsymbol{h}_k\|^2 + \|\boldsymbol{e}_k \times \boldsymbol{\nu}\|^{1+\alpha}_{\mathbf{L}_{1+\alpha}(\Gamma)} = (\boldsymbol{a}, \boldsymbol{e}_k) + (\boldsymbol{b}, \boldsymbol{h}_k). \tag{13}$$

Applying the Cauchy inequality we deduce that

$$\|\boldsymbol{e}_k\|^2 + \|\boldsymbol{h}_k\|^2 + \|\boldsymbol{e}_k \times \boldsymbol{\nu}\|^{1+\alpha}_{\mathbf{L}_{1+\alpha}(\Gamma)} \leqslant \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2.$$

Moreover we have

$$\|\boldsymbol{e}_k \times \boldsymbol{\nu}\|^{1+\alpha}_{\mathbf{L}_{1+\alpha}(\Gamma)} = \big\| |\boldsymbol{e}_k \times \boldsymbol{\nu}|^{\alpha-1} \boldsymbol{e}_k \times \boldsymbol{\nu} \big\|^{\frac{1+\alpha}{\alpha}}_{\mathbf{L}_{\frac{1+\alpha}{\alpha}}(\Gamma)} \leqslant \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2.$$

Setting $\boldsymbol{\varphi} = \boldsymbol{e}_k$ and $\boldsymbol{\psi} = \nabla \times \boldsymbol{e}_k$ in (9), we obtain

$$\|\boldsymbol{e}_k\|^2 + \|\nabla \times \boldsymbol{e}_k\|^2 + \|\boldsymbol{e}_k \times \boldsymbol{\nu}\|^{1+\alpha}_{\mathbf{L}_{1+\alpha}(\Gamma)} \leqslant \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2.$$

Due to the reflexivity of $\mathbf{L}_2(\Omega)$, $\mathbf{L}_{\frac{1+\alpha}{\alpha}}(\Gamma)$ and $\mathbf{H}(\mathbf{curl},\Omega)$ we can find subsequences of $\{\boldsymbol{e}_k\}$ and $\{\boldsymbol{h}_k\}$ (which we again denote by the same symbols) such that

$$\begin{aligned}
&\boldsymbol{e}_k \rightharpoonup \boldsymbol{e} \quad \text{in } \mathbf{L}_2(\Omega),\\
&\nabla\times\boldsymbol{e}_k \rightharpoonup \nabla\times\boldsymbol{e} \quad \text{in } \mathbf{L}_2(\Omega),\\
&\boldsymbol{h}_k \rightharpoonup \boldsymbol{h} \quad \text{in } \mathbf{L}_2(\Omega),\\
&\boldsymbol{\nu}\times\left(|\boldsymbol{e}_k\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{e}_k\times\boldsymbol{\nu}\right) \rightharpoonup \boldsymbol{\nu}\times(\boldsymbol{z}\times\boldsymbol{\nu}) \quad \text{in } \mathbf{L}_{\frac{1+\alpha}{\alpha}}(\Gamma).
\end{aligned}\tag{14}$$

Passing to the limit for $k\to\infty$ in (9) and using (8) we arrive at

$$\begin{aligned}
&(\boldsymbol{e},\boldsymbol{\varphi}) - (\boldsymbol{h},\nabla\times\boldsymbol{\varphi}) + (\boldsymbol{z}\times\boldsymbol{\nu},\boldsymbol{\varphi}\times\boldsymbol{\nu})_\Gamma = (\boldsymbol{a},\boldsymbol{\varphi}),\\
&(\boldsymbol{h},\boldsymbol{\psi}) + (\nabla\times\boldsymbol{e},\boldsymbol{\psi}) = (\boldsymbol{b},\boldsymbol{\psi})
\end{aligned}\tag{15}$$

for any $\boldsymbol{\varphi}\in\boldsymbol{V}_{1+\alpha}$, $\boldsymbol{\psi}\in\mathbf{L}_2(\Omega)$.

Further, using (13) and later (15) for $\boldsymbol{\varphi}=\boldsymbol{e}$ and $\boldsymbol{\psi}=\boldsymbol{h}$ we successively deduce

$$\begin{aligned}
\lim_{k\to\infty}\left(|\boldsymbol{e}_k\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{e}_k\times\boldsymbol{\nu},\boldsymbol{e}_k\times\boldsymbol{\nu}\right)_\Gamma &= \lim_{k\to\infty}\left[(\boldsymbol{a},\boldsymbol{e}_k)+(\boldsymbol{b},\boldsymbol{h}_k)\right] - \lim_{k\to\infty}\left[\|\boldsymbol{e}_k\|^2+\|\boldsymbol{h}_k\|^2\right]\\
&\leqslant (\boldsymbol{a},\boldsymbol{e})+(\boldsymbol{b},\boldsymbol{h}) - \|\boldsymbol{e}\|^2 - \|\boldsymbol{h}\|^2\\
&= (\boldsymbol{z}\times\boldsymbol{\nu},\boldsymbol{e}\times\boldsymbol{\nu})_\Gamma.
\end{aligned}\tag{16}$$

Now, we are going to use the Minty–Browder trick (cf. [19,20]) in order to show that $\boldsymbol{\nu}\times(\boldsymbol{z}\times\boldsymbol{\nu}) = \boldsymbol{\nu}\times(|\boldsymbol{e}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{e}\times\boldsymbol{\nu})$. We use the monotone structure of the non-linear operator $|\mathbf{x}|^{\alpha-1}\mathbf{x}$,

$$\left(|\boldsymbol{e}_k\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{e}_k\times\boldsymbol{\nu} - |\boldsymbol{u}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{u}\times\boldsymbol{\nu},\boldsymbol{e}_k\times\boldsymbol{\nu}-\boldsymbol{u}\times\boldsymbol{\nu}\right)_\Gamma \geqslant 0,\tag{17}$$

which is valid for any vector field $\boldsymbol{u}\in\mathbf{L}_{1+\alpha}(\Gamma)$.

Therefore, passing to the limit for $k\to\infty$ in (17) we obtain from (16)

$$\left(\boldsymbol{z}\times\boldsymbol{\nu} - |\boldsymbol{u}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{u}\times\boldsymbol{\nu},\boldsymbol{e}\times\boldsymbol{\nu}-\boldsymbol{u}\times\boldsymbol{\nu}\right)_\Gamma \geqslant 0.$$

Now, we set $\boldsymbol{u}=\boldsymbol{e}+\varepsilon\boldsymbol{w}$ for any $\boldsymbol{w}\in\mathbf{L}_{1+\alpha}(\Gamma)$ and any $\varepsilon>0$. We get

$$\left(\boldsymbol{z}\times\boldsymbol{\nu} - |(\boldsymbol{e}+\varepsilon\boldsymbol{w})\times\boldsymbol{\nu}|^{\alpha-1}(\boldsymbol{e}+\varepsilon\boldsymbol{w})\times\boldsymbol{\nu},\boldsymbol{w}\times\boldsymbol{\nu}\right)_\Gamma \leqslant 0.$$

Passing to $\varepsilon\to 0$ we can write

$$\left(\boldsymbol{\nu}\times\boldsymbol{z}\times\boldsymbol{\nu} - \boldsymbol{\nu}\times|\boldsymbol{e}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{e}\times\boldsymbol{\nu},\boldsymbol{w}\right)_\Gamma \leqslant 0.$$

Since this is valid for both $\boldsymbol{w}=\boldsymbol{z}$ and $\boldsymbol{w}=-\boldsymbol{z}$, we know that

$$\left(\boldsymbol{\nu}\times\boldsymbol{z}\times\boldsymbol{\nu} - \boldsymbol{\nu}\times|\boldsymbol{e}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{e}\times\boldsymbol{\nu},\boldsymbol{w}\right)_\Gamma = 0, \quad \forall\boldsymbol{w}\in\mathbf{L}_{1+\alpha}(\Gamma),$$

from which we deduce that

$$\boldsymbol{\nu}\times(\boldsymbol{z}\times\boldsymbol{\nu}) = \boldsymbol{\nu}\times\left(|\boldsymbol{e}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{e}\times\boldsymbol{\nu}\right)$$

a.e. in Γ. According to (15) we see that $\boldsymbol{e}$ and $\boldsymbol{h}$ solve (7).

Suppose that $(\boldsymbol{e},\boldsymbol{h})$ and $(\boldsymbol{e}',\boldsymbol{h}')$ be two solutions to (7). Then

$$\begin{aligned}
&\left(\boldsymbol{e}-\boldsymbol{e}',\boldsymbol{\varphi}\right) - \left(\boldsymbol{h}-\boldsymbol{h}',\nabla\times\boldsymbol{\varphi}\right) + \left(|\boldsymbol{e}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{e}\times\boldsymbol{\nu} - \left|\boldsymbol{e}'\times\boldsymbol{\nu}\right|^{\alpha-1}\boldsymbol{e}'\times\boldsymbol{\nu},\boldsymbol{\varphi}\times\boldsymbol{\nu}\right)_\Gamma = 0,\\
&\left(\boldsymbol{h}-\boldsymbol{h}',\boldsymbol{\psi}\right) + \left(\nabla\times\left(\boldsymbol{e}-\boldsymbol{e}'\right),\boldsymbol{\psi}\right) = 0.
\end{aligned}\tag{18}$$

Setting $\boldsymbol{\varphi}=\boldsymbol{e}-\boldsymbol{e}'$, $\boldsymbol{\psi}=\boldsymbol{h}-\boldsymbol{h}'$ and using (11) we easily see that

$$\left\|\boldsymbol{e}-\boldsymbol{e}'\right\|^2 + \left\|\boldsymbol{h}-\boldsymbol{h}'\right\|^2 \leqslant 0,$$

which implies the uniqueness of a solution. $\square$

3. A priori estimates

Next step is to derive suitable a priori estimates for $\boldsymbol{e}_i, \boldsymbol{h}_i$, $i = 1, \ldots, n$. We do it in a few steps.

Lemma 3.1. *Assume $\boldsymbol{E}_0, \boldsymbol{H}_0 \in \mathbf{L}_2(\Omega)$. Then there exists a positive constant C such that*

$$\|\boldsymbol{e}_j\|^2 + \|\boldsymbol{h}_j\|^2 + \sum_{i=1}^{j} \|\boldsymbol{e}_i - \boldsymbol{e}_{i-1}\|^2 + \sum_{i=1}^{j} \|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\|^2 + \sum_{i=1}^{j} \tau \|\boldsymbol{e}_i \times \boldsymbol{\nu}\|_{\mathbf{L}_{1+\alpha}(\Gamma)}^{1+\alpha} \leqslant C$$

holds for all $j = 1, \ldots, n$.

Proof. We set $\boldsymbol{\varphi} = \boldsymbol{e}_i$ and $\boldsymbol{\psi} = \boldsymbol{h}_i$ in (6), then we sum both equations and get

$$(\delta \boldsymbol{e}_i, \boldsymbol{e}_i) + \|\boldsymbol{e}_i\|^2 + (\delta \boldsymbol{h}_i, \boldsymbol{h}_i) + \|\boldsymbol{e}_i \times \boldsymbol{\nu}\|_{\mathbf{L}_{1+\alpha}(\Gamma)}^{1+\alpha} = 0.$$

We multiply this by τ and sum it up for $i = 1, \ldots, j$. Using Abel's summation for the left-hand side we deduce

$$\|\boldsymbol{e}_j\|^2 + \|\boldsymbol{h}_j\|^2 + \sum_{i=1}^{j} \|\boldsymbol{e}_i - \boldsymbol{e}_{i-1}\|^2 + \sum_{i=1}^{j} \|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\|^2 \leqslant C\big(\|\boldsymbol{E}_0\|^2 + \|\boldsymbol{H}_0\|^2\big). \qquad \square$$

For the next lemma we need that the Maxwell equations (5) are satisfied at the time $t = 0$. Thus, if we assume that $\boldsymbol{E}_0, \boldsymbol{H}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$ then we can define

$$\partial_t \boldsymbol{E}(0) := \nabla \times \boldsymbol{H}_0 - \boldsymbol{E}_0,$$
$$\partial_t \boldsymbol{H}(0) := -\nabla \times \boldsymbol{E}_0,$$

which will imply that (5) is satisfied for $t = 0$.

Further we define

$$\delta \boldsymbol{e}_0 := \partial_t \boldsymbol{E}(0), \qquad \delta \boldsymbol{h}_0 := \partial_t \boldsymbol{h}(0),$$

which means that also (6) will be satisfied for $i = 0$.

For the next a priori estimates, we will need the following technical lemma. The proof can be found in [21].

Lemma 3.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that $G(s) := g(s)s$ is monotonically increasing. Let Φ_G be the primitive function of G. Then for any vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ we have*

(i) $\Phi_G(|\boldsymbol{b}|) - \Phi_G(|\boldsymbol{a}|) \leqslant g(|\boldsymbol{b}|)\boldsymbol{b}(\boldsymbol{b} - \boldsymbol{a})$,
(ii) $\Phi_G(|\boldsymbol{b}|) - \Phi_G(|\boldsymbol{a}|) \geqslant g(|\boldsymbol{a}|)\boldsymbol{a}(\boldsymbol{b} - \boldsymbol{a})$.

Higher regularity of the initial data $\boldsymbol{E}_0, \boldsymbol{H}_0$ will imply better regularity of $\boldsymbol{e}_j, \boldsymbol{h}_j$.

Lemma 3.3. *Assume $\boldsymbol{E}_0, \boldsymbol{H}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$. Then there exists a positive C such that (for any $j = 1, \ldots, n$)*

$$\|\delta \boldsymbol{e}_j\|^2 + \sum_{i=1}^{j} \|\delta \boldsymbol{e}_i - \delta \boldsymbol{e}_{i-1}\|^2 + \|\delta \boldsymbol{h}_j\|^2 + \sum_{i=1}^{j} \|\delta \boldsymbol{h}_i - \delta \boldsymbol{h}_{i-1}\|^2 \leqslant C.$$

Proof. We subtract (6) for $i = i - 1$ from (6), then we set $\boldsymbol{\varphi} = \delta \boldsymbol{e}_i$, $\boldsymbol{\psi} = \delta \boldsymbol{h}_i$ and we sum up both equations. We have

$$\begin{aligned}&(\delta \boldsymbol{e}_i - \delta \boldsymbol{e}_{i-1}, \delta \boldsymbol{e}_i) + (\boldsymbol{e}_i - \boldsymbol{e}_{i-1}, \delta \boldsymbol{e}_i) + (\delta \boldsymbol{h}_i - \delta \boldsymbol{h}_{i-1}, \delta \boldsymbol{h}_i)\\ &\quad \big(|\boldsymbol{e}_i \times \boldsymbol{\nu}|^{\alpha-1} \boldsymbol{e}_i \times \boldsymbol{\nu} - |\boldsymbol{e}_{i-1} \times \boldsymbol{\nu}|^{\alpha-1} \boldsymbol{e}_{i-1} \times \boldsymbol{\nu}, \delta \boldsymbol{e}_i \times \boldsymbol{\nu}\big)_\Gamma = 0.\end{aligned} \tag{19}$$

The boundary term is positive due to monotonicity of the non-linearity. We sum up the expression for $i = 1, \ldots, j$ and apply the Abel summation for the first and third term on the left and we deduce

$$\begin{aligned}\|\delta \boldsymbol{e}_j\|^2 + \sum_{i=1}^{j} \|\delta \boldsymbol{e}_i - \delta \boldsymbol{e}_{i-1}\|^2 + \|\delta \boldsymbol{h}_j\|^2 + \sum_{i=1}^{j} \|\delta \boldsymbol{h}_i - \delta \boldsymbol{h}_{i-1}\|^2 &\leqslant C\big(\|\delta \boldsymbol{e}_0\|^2 + \|\delta \boldsymbol{h}_0\|^2\big)\\ &\leqslant C\big(\|\boldsymbol{E}_0\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 + \|\boldsymbol{H}_0\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\big). \qquad \square\end{aligned}$$

Lemma 3.4. *Assume $\boldsymbol{E}_0, \boldsymbol{H}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$. Then there exists a positive C such that (for any $j = 1, \dots, n$)*

$$\|\delta \boldsymbol{e}_j\|^2 + \sum_{i=1}^{j} \|\delta \boldsymbol{e}_i - \delta \boldsymbol{e}_{i-1}\|^2 + \|\nabla \times \boldsymbol{e}_j\|^2 + \sum_{i=1}^{j} \big\|\nabla \times (\boldsymbol{e}_i - \boldsymbol{e}_{i-1})\big\|^2 \leqslant C.$$

Proof. We subtract (6a) for $i = i - 1$ from (6a) and set $\boldsymbol{\varphi} = \delta \boldsymbol{e}_i$. In (6b) we set $\boldsymbol{\psi} = \nabla \times \delta \boldsymbol{e}_i$ and add it to the previous expression. If we write the non-linearity as (10), we get

$$(\delta \boldsymbol{e}_i - \delta \boldsymbol{e}_{i-1}, \delta \boldsymbol{e}_i) + (\boldsymbol{e}_i - \boldsymbol{e}_{i-1}, \delta \boldsymbol{e}_i) + \big(\nabla \times \boldsymbol{e}_i, \nabla \times (\boldsymbol{e}_i - \boldsymbol{e}_{i-1})\big) + \tau \big(\delta \boldsymbol{a}(\boldsymbol{e}_i \times \boldsymbol{\nu}), \delta \boldsymbol{e}_i \times \boldsymbol{\nu}\big)_\Gamma = 0.$$

After taking the sum for $i = 1, \dots, j$, the estimate follows from Abel's summation rule in the same way as for the proof of Lemma 3.3. $\square$

The previous lemma's admit to obtain also an estimate for the curl of the magnetic field.

Lemma 3.5. *Let the assumptions from Lemma* 3.4 *be satisfied. Then there exists a positive constant C, such that for any $i = 1, \dots, n$,*

$$\|\nabla \times \boldsymbol{h}_i\| \leqslant C.$$

Proof. From (6a) we know that for every $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\overline{\Omega})$,

$$\begin{aligned}\big|(\nabla \times \boldsymbol{h}_i, \boldsymbol{\varphi})\big| &= \big|(\delta \boldsymbol{e}_i + \boldsymbol{e}_i, \boldsymbol{\varphi})\big| \\ &\leqslant \big(\|\boldsymbol{e}_i\| + \|\delta \boldsymbol{e}_i\|\big)\|\boldsymbol{\varphi}\| \\ &\leqslant C\|\boldsymbol{\varphi}\|,\end{aligned}$$

where we used Lemma 3.1 and 3.3. From the density of $\mathbf{C}_0^\infty(\overline{\Omega})$ in $\mathbf{L}_2(\Omega)$, we conclude that $\|\nabla \times \boldsymbol{h}_i\| \leqslant C$, where C does not depend on i. $\square$

4. Convergence

In this section we prove the convergence of our approximate solution to a weak solution of (5) in suitable function spaces.

First, we introduce the piecewise linear in time vector field $\boldsymbol{h}_n$ $(i = 1, \dots, n)$ given by

$$\begin{aligned}&\boldsymbol{h}_n(0) = \boldsymbol{H}_0, \\ &\boldsymbol{h}_n(t) = \boldsymbol{h}_{i-1} + (t - t_{i-1})\delta \boldsymbol{h}_i \quad \text{for } t \in (t_{i-1}, t_i].\end{aligned}$$

Next, we define the piecewise constant vector field $\bar{\boldsymbol{h}}_n$,

$$\begin{aligned}&\bar{\boldsymbol{h}}_n(0) = \boldsymbol{H}_0, \\ &\bar{\boldsymbol{h}}_n(t) = \boldsymbol{h}_i \quad \text{for } t \in (t_{i-1}, t_i].\end{aligned}$$

Similarly we define the time-dependent vector fields $\boldsymbol{e}_n$ and $\bar{\boldsymbol{e}}_n$.

Using the new notation we rewrite (6) as

$$\begin{aligned}&(\partial_t \boldsymbol{e}_n, \boldsymbol{\varphi}) + (\bar{\boldsymbol{e}}_n, \boldsymbol{\varphi}) - (\bar{\boldsymbol{h}}_n, \nabla \times \boldsymbol{\varphi}) + \big(|\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\alpha-1} \bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}, \boldsymbol{\varphi} \times \boldsymbol{\nu}\big)_\Gamma = 0, \\ &(\partial_t \boldsymbol{h}_n, \boldsymbol{\psi}) + (\nabla \times \bar{\boldsymbol{e}}_n, \boldsymbol{\psi}) = 0,\end{aligned} \tag{20}$$

with $\boldsymbol{\varphi} \in \boldsymbol{V}_{1+\alpha}$ and $\boldsymbol{\psi} \in \mathbf{L}_2(\Omega)$.

Our next step is to show the existence of a weak solution of (5). To do this, we will use the stability results of previous lemmas. The next theorem is valid for subsequences from $\{\boldsymbol{e}_n\}$ and $\{\boldsymbol{h}_n\}$, which will be denoted by the same symbol again.

Theorem 4.1 (Convergence). *Let $\boldsymbol{E}_0, \boldsymbol{H}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$. Then there exist fields*

$$\boldsymbol{E}, \boldsymbol{H} \in L_2\big((0,T), \mathbf{H}(\mathbf{curl}, \Omega)\big) \cap H^1\big((0,T), \mathbf{L}_2(\Omega)\big),$$

with

$$\boldsymbol{E} \times \boldsymbol{\nu} \in L_{1+\alpha}\big((0,T), \mathbf{L}_{1+\alpha}(\Gamma)\big),$$

such that

(i) $\bar{\boldsymbol{e}}_n \rightharpoonup \boldsymbol{E}$ *in* $L_2((0,T),\mathbf{H}(\mathbf{curl},\Omega))$ *and* $\partial_t \boldsymbol{e}_n \rightharpoonup \partial_t \boldsymbol{E}$ *in* $L_2((0,T),\mathbf{L}_2(\Omega))$.
(ii) $\bar{\boldsymbol{h}}_n \rightharpoonup \boldsymbol{H}$ *in* $L_2((0,T),\mathbf{H}(\mathbf{curl},\Omega))$ *and* $\partial_t \boldsymbol{h}_n \rightharpoonup \partial_t \boldsymbol{H}$ *in* $L_2((0,T),\mathbf{L}_2(\Omega))$.
(iii) $|\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\alpha-1}\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu} \rightharpoonup |\boldsymbol{E}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E}\times\boldsymbol{\nu}$ *in* $L_{\frac{\alpha+1}{\alpha}}((0,T),\mathbf{L}_{\frac{\alpha+1}{\alpha}})$.
(iv) $\boldsymbol{E},\boldsymbol{H}$ *solve* (5).

Proof. (i) Lemma's 3.1 and 3.4 imply that $\int_0^T (\|\bar{\boldsymbol{e}}_n(t)\|^2 + \|\nabla\times\bar{\boldsymbol{e}}_n(t)\|^2) < C$. From Lemma 3.3 we have $\int_0^T \|\partial_t \boldsymbol{e}_n(t)\|^2 < C$. The statement follows from the weak compactness property of the corresponding spaces.

(ii) This follows the same reasoning as (i).

(iii) From Lemma 3.1 we obtain that

$$\int_0^T \big\| |\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\alpha-1}\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}\big\|^{\frac{\alpha+1}{\alpha}}_{\mathbf{L}_{\frac{\alpha+1}{\alpha}}(\Gamma)} \leqslant C,$$

i.e.,

$$|\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\alpha-1}\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu} \rightharpoonup \boldsymbol{z},$$

for some $\boldsymbol{z} \in L_{\frac{1+\alpha}{\alpha}}((0,T),\mathbf{L}_{\frac{1+\alpha}{\alpha}}(\Gamma))$. To prove that $\boldsymbol{z} = |\boldsymbol{E}\times\boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E}\times\boldsymbol{\nu}$, we apply again the Minty–Browder argument. First, we integrate (20) in time for any $t \in (0,T)$ and then we pass to the limit for $n \to \infty$. We get

$$\begin{aligned}
&\int_0^t (\partial_t \boldsymbol{E},\boldsymbol{\varphi}) + \int_0^t (\boldsymbol{E},\boldsymbol{\varphi}) - \int_0^t (\boldsymbol{H},\nabla\times\boldsymbol{\varphi}) + \int_0^t (\boldsymbol{z},\boldsymbol{\varphi}\times\boldsymbol{\nu})_\Gamma = 0,\\
&\int_0^t (\partial_t \boldsymbol{H},\boldsymbol{\psi}) + \int_0^t (\nabla\times\boldsymbol{E},\boldsymbol{\psi}) = 0.
\end{aligned} \tag{21}$$

Now, we set $\boldsymbol{\varphi} = \bar{\boldsymbol{e}}_n$ and $\boldsymbol{\psi} = \bar{\boldsymbol{h}}_n$ in (20), sum up both equations and integrate the result in time. We also employ the fact that $a_n \rightharpoonup a$ in $L_2(\Omega)$ implies $\lim_{n\to\infty}\|a_n\|^2 \geqslant \|a\|^2$. We successively deduce

$$\begin{aligned}
\lim_{n\to\infty}\int_0^t\int_\Gamma |\bar{\boldsymbol{e}}_n\times\boldsymbol{\nu}|^{1+\alpha} &\overset{(20)}{=} -\lim_{n\to\infty}\int_0^t \big[(\partial_t \boldsymbol{e}_n,\bar{\boldsymbol{e}}_n) + (\partial_t \boldsymbol{h}_n,\bar{\boldsymbol{h}}_n) + \|\bar{\boldsymbol{e}}_n\|^2\big]\\
&= -\lim_{n\to\infty}\int_0^t \big[(\partial_t \boldsymbol{e}_n,\boldsymbol{e}_n) + (\partial_t \boldsymbol{h}_n,\boldsymbol{h}_n) + \|\boldsymbol{e}_n\|^2\big]\\
&= \frac{\|\boldsymbol{E}_0\|^2 + \|\boldsymbol{H}_0\|^2}{2} - \lim_{n\to\infty}\left[\frac{\|\boldsymbol{e}_n(t)\|^2 + \|\boldsymbol{h}_n(t)\|^2}{2} + \int_0^t \|\boldsymbol{e}_n\|^2\right]\\
&\leqslant \frac{\|\boldsymbol{E}_0\|^2 + \|\boldsymbol{H}_0\|^2}{2} - \frac{\|\boldsymbol{E}(t)\|^2 + \|\boldsymbol{H}(t)\|^2}{2} - \int_0^t \|\boldsymbol{E}\|^2\\
&= -\int_0^t \big[(\partial_t \boldsymbol{E},\boldsymbol{E}) + (\partial_t \boldsymbol{H},\boldsymbol{H}) + \|\boldsymbol{E}\|^2\big]\\
&\overset{(21)}{=} \int_0^t (\boldsymbol{z},\boldsymbol{E}\times\boldsymbol{\nu})_\Gamma.
\end{aligned} \tag{22}$$

From the monotonicity of the non-linear operator (cf. (11)) we have

$$\int_0^t \big(|\bar{\boldsymbol{e}}_n\times\boldsymbol{\nu}|^{\alpha-1}\bar{\boldsymbol{e}}_n\times\boldsymbol{\nu} - |\boldsymbol{u}|^{\alpha-1}\boldsymbol{u},\bar{\boldsymbol{e}}_n\times\boldsymbol{\nu} - \boldsymbol{u}\big)_\Gamma \geqslant 0, \tag{23}$$

which is valid for any vector field $\boldsymbol{u} \in L_{1+\alpha}((0,T),\mathbf{L}_{1+\alpha}(\Gamma))$.

Therefore, passing to the limit for $n \to \infty$ in (23) we obtain

$$\int_0^t \big(\boldsymbol{z} - |\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}, \boldsymbol{E} \times \boldsymbol{\nu} - \boldsymbol{u}\big)_\Gamma \geqslant 0.$$

Now, we set $\boldsymbol{u} = \boldsymbol{E} \times \boldsymbol{\nu} + \varepsilon \boldsymbol{w}$ for any $\boldsymbol{w} \in L_{1+\alpha}((0,T), \mathbf{L}_{1+\alpha}(\Gamma))$ and $\varepsilon > 0$,

$$\int_0^t \big(\boldsymbol{z} - |\boldsymbol{E} \times \boldsymbol{\nu} + \varepsilon \boldsymbol{w}|^{\alpha-1}(\boldsymbol{E} \times \boldsymbol{\nu} + \varepsilon \boldsymbol{w}), \boldsymbol{w}\big)_\Gamma \leqslant 0.$$

Passing to $\varepsilon \to 0$ we can write

$$\int_0^t \big(\boldsymbol{z} - |\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu}, \boldsymbol{w}\big)_\Gamma \leqslant 0.$$

Since the inequality is valid for both $\boldsymbol{w} = \boldsymbol{z}$ and $\boldsymbol{w} = -\boldsymbol{z}$, we conclude

$$\int_0^t \big(\boldsymbol{z} - |\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu}, \boldsymbol{w}\big)_\Gamma = 0, \quad \forall \boldsymbol{w} \in L_{1+\alpha}\big((0,T), \mathbf{L}_{1+\alpha}(\Gamma)\big),$$

from which we conclude $\boldsymbol{z} = |\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu}$ a.e. in $(0,T) \times \Gamma$.

(iv) According to (21) we see that

$$\int_0^t (\partial_t \boldsymbol{E}, \boldsymbol{\varphi}) + \int_0^t (\boldsymbol{E}, \boldsymbol{\varphi}) - \int_0^t (\boldsymbol{H}, \nabla \times \boldsymbol{\varphi}) + \int_0^t \big(|\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \times \boldsymbol{\nu}\big)_\Gamma = 0,$$

$$\int_0^t (\partial_t \boldsymbol{H}, \boldsymbol{\psi}) + \int_0^t (\nabla \times \boldsymbol{E}, \boldsymbol{\psi}) = 0.$$

This is valid for any $t \in (0,T)$. After differentiation with respect to time, we see that $\boldsymbol{E}$ and $\boldsymbol{H}$ solve (5). □

The following theorem derives the error estimates for the time discretization method.

Theorem 4.2 *(Error estimates). Let $\boldsymbol{E}_0, \boldsymbol{H}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$. Then there exists a positive constant C such that*

$$\max_{t\in[0,T]} \big\|\boldsymbol{e}_n(t) - \boldsymbol{E}(t)\big\|^2 + \max_{t\in[0,T]} \big\|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\big\|^2 + \int_0^T \int_\Gamma \big[|\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\frac{\alpha+1}{2}} - |\boldsymbol{E} \times \boldsymbol{\nu}|^{\frac{\alpha+1}{2}}\big]^2 \leqslant C\tau.$$

Proof. We subtract (20) from (5). Then we set $\boldsymbol{\varphi} = \bar{\boldsymbol{e}}_n - \boldsymbol{E}$ and $\boldsymbol{\psi} = \bar{\boldsymbol{h}}_n - \boldsymbol{H}$. After integration over $(0,t)$ for any $t \in [0,T]$ we obtain

$$\begin{aligned}
&\int_0^t \big(\partial_t(\boldsymbol{e}_n - \boldsymbol{E}), \boldsymbol{e}_n - \boldsymbol{E}\big) + \int_0^t \|\bar{\boldsymbol{e}}_n - \boldsymbol{E}\|^2 + \int_0^t \big(\partial_t(\boldsymbol{h}_n - \boldsymbol{H}), \boldsymbol{h}_n - \boldsymbol{H}\big)\\
&\quad + \int_0^t \big(|\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\alpha-1}\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu} - |\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu}, (\bar{\boldsymbol{e}}_n - \boldsymbol{E}) \times \boldsymbol{\nu}\big)_\Gamma\\
&= \int_0^t \big(\partial_t(\boldsymbol{e}_n - \boldsymbol{E}), \boldsymbol{e}_n - \bar{\boldsymbol{e}}_n\big) + \int_0^t \big(\partial_t(\boldsymbol{h}_n - \boldsymbol{H}), \boldsymbol{h}_n - \bar{\boldsymbol{h}}_n\big).
\end{aligned} \tag{24}$$

Using the stability results from Lemma 3.3, the right-hand side of (24) can be estimated as follows

$$\int_0^t \big(\partial_t(\boldsymbol{e}_n - \boldsymbol{E}), \boldsymbol{e}_n - \bar{\boldsymbol{e}}_n\big) \leqslant C\tau \int_0^t \big(\|\partial_t \boldsymbol{e}_n\| + \|\partial_t \boldsymbol{E}\|\big)\|\partial_t \boldsymbol{e}_n\| \leqslant C\tau$$

and

$$\int_0^t \left(\partial_t(\boldsymbol{h}_n - \boldsymbol{H}), \boldsymbol{h}_n - \bar{\boldsymbol{h}}_n\right) \leqslant C\tau \int_0^t \left(\|\partial_t \boldsymbol{h}_n\| + \|\partial_t \boldsymbol{H}\|\right)\|\partial_t \boldsymbol{h}_n\| \leqslant C\tau.$$

We shall use the following algebraic inequality, which can be proved in a standard way and which is valid for any $a, b, y, z \geqslant 0$,

$$4ab\left(y^{\frac{a+b}{2}} - z^{\frac{a+b}{2}}\right)^2 \leqslant (a+b)^2\left(y^a - z^a\right)\left(y^b - z^b\right). \tag{25}$$

Using (25) and the Cauchy inequality, we deduce

$$\begin{aligned}
\left(|\boldsymbol{y}|^{\alpha-1}\boldsymbol{y} - |\boldsymbol{z}|^{\alpha-1}\boldsymbol{z}\right)(\boldsymbol{y} - \boldsymbol{z}) &= |\boldsymbol{y}|^{\alpha+1} + |\boldsymbol{z}|^{\alpha+1} - |\boldsymbol{z}|^{\alpha-1}\boldsymbol{z}\boldsymbol{y} - |\boldsymbol{y}|^{\alpha-1}\boldsymbol{z}\boldsymbol{y} \\
&\geqslant |\boldsymbol{y}|^{\alpha+1} + |\boldsymbol{z}|^{\alpha+1} - |\boldsymbol{z}|^{\alpha}|\boldsymbol{y}| - |\boldsymbol{y}|^{\alpha}|\boldsymbol{z}| \\
&= \left(|\boldsymbol{y}|^{\alpha} - |\boldsymbol{z}|^{\alpha}\right)(|\boldsymbol{y}| - |\boldsymbol{z}|) \\
&\geqslant \frac{4\alpha}{(\alpha+1)^2}\left(|\boldsymbol{y}|^{\frac{\alpha+1}{2}} - |\boldsymbol{z}|^{\frac{\alpha+1}{2}}\right)^2.
\end{aligned}$$

Therefore, the boundary term in (24) can be estimated from below as follows

$$\int_0^t \left(|\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\alpha-1}\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu} - |\boldsymbol{E} \times \boldsymbol{\nu}|^{\alpha-1}\boldsymbol{E} \times \boldsymbol{\nu}, \bar{\boldsymbol{e}}_n \times \boldsymbol{\nu} - \boldsymbol{E} \times \boldsymbol{\nu}\right)_\Gamma \geqslant \frac{4\alpha}{(\alpha+1)^2}\int_0^t \int_\Gamma \left[|\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\frac{\alpha+1}{2}} - |\boldsymbol{E} \times \boldsymbol{\nu}|^{\frac{\alpha+1}{2}}\right]^2.$$

Collecting all estimates we arrive at

$$\left\|\boldsymbol{e}_n(t) - \boldsymbol{E}(t)\right\|^2 + \left\|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\right\|^2 + \int_0^t \int_\Gamma \left[|\bar{\boldsymbol{e}}_n \times \boldsymbol{\nu}|^{\frac{\alpha+1}{2}} - |\boldsymbol{E} \times \boldsymbol{\nu}|^{\frac{\alpha+1}{2}}\right]^2 \leqslant C\tau,$$

which is valid for all $t \in (0, T)$. $\square$

Finally, let us note that Theorem 4.2 also implies the uniqueness of the solution to (5).

5. Full discretization

In the previous section, we have proved convergence of backward Euler's method to the unique solution of (5). Now we study a fully discrete approximation scheme, based on conforming finite elements. We introduce the standard finite element space for lowest order Nédélec elements on tetrahedra,

$$\mathbf{V}_h = \left\{\boldsymbol{\varphi} \in \mathbf{H}(\mathbf{curl}, \Omega) \mid \boldsymbol{\varphi}|_K = \mathbf{a} + \mathbf{b} \times \mathbf{x},\ \forall K \in \mathcal{T}_h\right\}.$$

The corresponding degrees of freedom are the circulations along the edges of the mesh. The $\mathbf{L}_2(\Omega)$-conforming finite element space consists of piecewise constant fields on every tetrahedron and is denoted by $\mathbf{L}_h$ [15].

The edge element interpolation operator Π^h is defined such that $\Pi^h\mathbf{u} \in \mathbf{V}_h$ has the same degrees of freedom as $\mathbf{u}$. The largest space for which the circulations along the edges are well defined is the space of functions $\mathbf{u} \in \mathbf{L}_p(\Omega)$, with $\nabla \times \mathbf{u} \in \mathbf{L}_p(\Omega)$ and $\mathbf{u} \times \mathbf{n} \in \mathbf{L}_p(\partial\Omega)$, with $p > 2$ [22]. The approximation properties of this operator in both the $\mathbf{L}_2(\Omega)$- and the $\mathbf{H}(\mathbf{curl}, \Omega)$-norm are well documented (see e.g. [13,12]). In order to apply these estimates, we will restrict the interpolation operator to the space $\mathbf{H}^1(\mathbf{curl}, \Omega)$ of functions $\mathbf{u} \in \mathbf{H}^1(\Omega)$ for which $\nabla \times \mathbf{u} \in \mathbf{H}^1(\Omega)$. Finally we also introduce the orthogonal projection operator $P^h : \mathbf{L}^2(\Omega) \to \mathbf{L}_h$.

After these preliminaries, we can formulate the fully discrete approximation scheme,

$$\begin{aligned}
&\left(\delta\boldsymbol{e}_i^h, \boldsymbol{\varphi}^h\right) + \left(\boldsymbol{e}_i^h, \boldsymbol{\varphi}^h\right) - \left(\boldsymbol{h}_i^h, \nabla \times \boldsymbol{\varphi}^h\right) + \left(\left|\boldsymbol{e}_i^h \times \boldsymbol{\nu}\right|^{\alpha-1}\boldsymbol{e}_i^h \times \boldsymbol{\nu}, \boldsymbol{\varphi}^h \times \boldsymbol{\nu}\right)_\Gamma = 0, \\
&\left(\delta\boldsymbol{h}_i^h, \boldsymbol{\psi}^h\right) + \left(\nabla \times \boldsymbol{e}_i^h, \boldsymbol{\psi}^h\right) = 0, \\
&\boldsymbol{h}_0^h = P^h\boldsymbol{H}_0, \\
&\boldsymbol{e}_0^h = \Pi^h\boldsymbol{E}_0
\end{aligned} \tag{26}$$

for every $\boldsymbol{\varphi}^h \in \mathbf{V}_h$ and $\boldsymbol{\psi}^h \in \mathbf{L}_h$. Existence and uniqueness of the fields $\boldsymbol{e}_i^h$ and $\boldsymbol{h}_i^h$ were proved in Lemma 2.2 for general finite dimensional subspaces of $\boldsymbol{V}_{1+\alpha}$ and $\mathbf{L}_2(\Omega)$. With these fields, we can again construct the fields $\boldsymbol{e}_n^h$ and $\bar{\boldsymbol{e}}_n^h$ by piecewise linear and piecewise constant interpolation (see also Section 4), which will satisfy the same estimates as Lemma's 3.1, 3.3 and 3.4. The fully discretized system can thus be formulated as

$$\begin{aligned}&\left(\partial_t \boldsymbol{e}_n^h, \boldsymbol{\varphi}^h\right) + \left(\bar{\boldsymbol{e}}_n^h, \boldsymbol{\varphi}^h\right) - \left(\bar{\boldsymbol{h}}_n^h, \nabla \times \boldsymbol{\varphi}^h\right) + \left(\left|\bar{\boldsymbol{e}}_n^h \times \boldsymbol{\nu}\right|^{\alpha-1} \bar{\boldsymbol{e}}_n^h \times \boldsymbol{\nu}, \boldsymbol{\varphi}^h \times \boldsymbol{\nu}\right)_\Gamma = 0,\\ &\left(\partial_t \boldsymbol{h}_n^h, \boldsymbol{\psi}^h\right) + \left(\nabla \times \bar{\boldsymbol{e}}_n^h, \boldsymbol{\psi}^h\right) = 0.\end{aligned} \tag{27}$$

After subtracting (27) from (20), we put $\boldsymbol{\varphi}^h = \bar{\boldsymbol{e}}_n^h - \Pi^h \boldsymbol{E}$, $\boldsymbol{\psi}^h = \bar{\boldsymbol{h}}_n^h - P^h \boldsymbol{H}$, leaving

$$\begin{aligned}&\left(\partial_t\left(\boldsymbol{e}_n^h - \boldsymbol{E}\right), \bar{\boldsymbol{e}}_n^h - \Pi^h \boldsymbol{E}\right) - \left(\bar{\boldsymbol{h}}_n^h - \boldsymbol{H}, \nabla \times \left(\bar{\boldsymbol{e}}_n^h - \Pi^h \boldsymbol{E}\right)\right) + \left(\bar{\boldsymbol{e}}_n^h - \boldsymbol{E}, \bar{\boldsymbol{e}}_n^h - \Pi^h \boldsymbol{E}\right)\\ &\quad + \left(\boldsymbol{a}\left(\bar{\boldsymbol{e}}_n^h \times \boldsymbol{\nu}\right) - \boldsymbol{a}(\boldsymbol{E} \times \boldsymbol{\nu}), \left(\bar{\boldsymbol{e}}_n^h - \Pi^h \boldsymbol{E}\right) \times \boldsymbol{\nu}\right)_\Gamma = 0,\\ &\left(\partial_t\left(\boldsymbol{h}_n^h - \boldsymbol{H}\right), \bar{\boldsymbol{h}}_n^h - P^h \boldsymbol{H}\right) + \left(\nabla \times \left(\bar{\boldsymbol{e}}_n^h - \boldsymbol{E}\right), \bar{\boldsymbol{h}}_n^h - P^h \boldsymbol{H}\right) = 0,\end{aligned}$$

where we applied again the notation (10) for the non-linearity. We add both expressions and integrate in time. After some rearrangements, we obtain

$$\begin{aligned}&\frac{1}{2}\left\|\boldsymbol{e}_n^h(t) - \boldsymbol{E}(t)\right\|^2 - \frac{1}{2}\left\|\Pi^h \boldsymbol{E}_0 - \boldsymbol{E}_0\right\|^2 + \int_0^t \left\|\bar{\boldsymbol{e}}_n^h - \boldsymbol{E}\right\|^2 + \frac{1}{2}\left\|\boldsymbol{h}_n^h(t) - \boldsymbol{H}(t)\right\|^2 - \frac{1}{2}\left\|P^h \boldsymbol{H}_0 - \boldsymbol{H}_0\right\|^2\\ &\quad = \int_0^t \left(\partial_t\left(\boldsymbol{e}_n^h - \boldsymbol{E}\right), \boldsymbol{e}_n^h - \bar{\boldsymbol{e}}_n^h\right) + \int_0^t \left(\partial_t\left(\boldsymbol{e}_n^h - \boldsymbol{E}\right), \Pi^h \boldsymbol{E} - \boldsymbol{E}\right) + \int_0^t \left(\bar{\boldsymbol{e}}_n^h - \boldsymbol{E}, \Pi^h \boldsymbol{E} - \boldsymbol{E}\right)\\ &\qquad + \int_0^t \left(\partial_t\left(\boldsymbol{h}_n^h - \boldsymbol{H}\right), \bar{\boldsymbol{h}}_n^h - \boldsymbol{h}_n^h\right) + \int_0^t \left(\partial_t\left(\boldsymbol{h}_n^h - \boldsymbol{H}\right), P^h \boldsymbol{H} - \boldsymbol{H}\right) - \int_0^t \left(\nabla \times \left(\bar{\boldsymbol{e}}_n^h - \boldsymbol{E}\right), \boldsymbol{H} - P^h \boldsymbol{H}\right)\\ &\qquad + \int_0^t \left(\bar{\boldsymbol{h}}_n^h - \boldsymbol{H}, \nabla \times \left(\boldsymbol{E} - \Pi^h \boldsymbol{E}\right)\right) + \int_0^t \left(\boldsymbol{a}\left(\bar{\boldsymbol{e}}_n^h \times \boldsymbol{\nu}\right) - \boldsymbol{a}(\boldsymbol{E} \times \boldsymbol{\nu}), \Pi^h \boldsymbol{E} \times \boldsymbol{\nu} - \boldsymbol{E} \times \boldsymbol{\nu}\right)_\Gamma\\ &\quad := \sum_{i=1}^8 S_i.\end{aligned}$$

We will now derive upper bounds for the terms S_i, $i = 1, \ldots, 8$. For S_1 and S_4 we apply Cauchy's inequality,

$$\begin{aligned}&S_1 \leqslant \tau \sqrt{\int_0^t \left\|\partial_t \boldsymbol{e}_n^h(t)\right\|^2} \cdot \sqrt{\int_0^t \left\|\partial_t \boldsymbol{e}_n^h - \partial_t \boldsymbol{E}\right\|^2} \leqslant C\tau,\\ &S_4 \leqslant C\tau.\end{aligned}$$

The terms S_2 and S_5 can be bounded in the same way. If we assume $\boldsymbol{E} \in L_2((0,T), \mathbf{H}^1(\mathbf{curl}, \Omega))$, then we obtain from the approximation properties of Π^h [12,17] and P^h [23],

$$\begin{aligned}&S_2 \leqslant C\sqrt{\int_0^t \left\|\Pi^h \boldsymbol{E} - \boldsymbol{E}\right\|^2} \leqslant Ch\sqrt{\int_0^t \|\boldsymbol{E}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)}},\\ &S_5 \leqslant Ch\sqrt{\int_0^t \|\boldsymbol{H}\|^2_{\mathbf{H}^1(\Omega)}},\end{aligned}$$

since we have proved that the weak solution satisfies $\boldsymbol{H} \in C^1([0,T], \mathbf{L}^2(\Omega))$ (Theorem 4.2). For the third term we get, based on the triangle inequality,

$$S_3 \leqslant C\tau^2 + C\int_0^t \left\|\boldsymbol{e}_n^h - \boldsymbol{E}\right\|^2 + Ch^2 \int_0^t \|\boldsymbol{E}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)}.$$

For $S_6 + S_7$, we find after some arrangements,

$$\begin{aligned}
S_6 + S_7 &= -\int_0^t \left(\bar{\boldsymbol{h}}_n^h - \boldsymbol{H}, \nabla \times \left(\Pi^h \boldsymbol{E} - \boldsymbol{E}\right)\right) - \int_0^t \left(\nabla \times \left(\bar{\boldsymbol{e}}_n^h - \boldsymbol{E}\right), \boldsymbol{H} - P^h \boldsymbol{H}\right) \\
&\leqslant C\tau^2 + C\int_0^t \left\|\boldsymbol{h}_n^h - \boldsymbol{H}\right\|^2 + C\int_0^t \left\|\nabla \times \left(\Pi^h \boldsymbol{E} - \boldsymbol{E}\right)\right\|^2 + C\sqrt{\int_0^t \left\|\boldsymbol{H} - P^h \boldsymbol{H}\right\|^2} \\
&\leqslant C\tau^2 + C\int_0^t \left\|\boldsymbol{h}_n^h - \boldsymbol{H}\right\|^2 + Ch^2\int_0^t \|\boldsymbol{E}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)} + Ch\sqrt{\int_0^t \|\boldsymbol{H}\|^2_{\mathbf{H}^1(\Omega)}},
\end{aligned}$$

where we have applied Cauchy's and Young's inequality. Finally the boundary integral can be bounded for $0 < \alpha \leqslant 1$, using the continuous imbedding of $\mathbf{L}^2(\Gamma)$ in $\mathbf{L}^{\alpha+1}(\Gamma)$,

$$\begin{aligned}
S_8 &\leqslant \int_0^t \left\|\boldsymbol{a}\left(\bar{\boldsymbol{e}}_n^h \times \boldsymbol{\nu}\right) - \boldsymbol{a}(\boldsymbol{E} \times \boldsymbol{\nu})\right\|_{\frac{\alpha+1}{\alpha}} \cdot \left\|\Pi^h \boldsymbol{E} \times \boldsymbol{\nu} - \boldsymbol{E} \times \boldsymbol{\nu}\right\|_{\alpha+1} \\
&\leqslant \left(\int_0^t \left\|\boldsymbol{a}\left(\bar{\boldsymbol{e}}_n^h \times \boldsymbol{\nu}\right) - \boldsymbol{a}(\boldsymbol{E} \times \boldsymbol{\nu})\right\|^{\frac{\alpha+1}{\alpha}}_{\frac{\alpha+1}{\alpha}}\right)^{\frac{\alpha}{\alpha+1}} \left(\int_0^t \left\|\Pi^h \boldsymbol{E} \times \boldsymbol{\nu} - \boldsymbol{E} \times \boldsymbol{\nu}\right\|^{\alpha+1}_{\alpha+1}\right)^{\frac{1}{\alpha+1}} \\
&\leqslant C\left(\int_0^t \left\|\Pi^h \boldsymbol{E} \times \boldsymbol{\nu} - \boldsymbol{E} \times \boldsymbol{\nu}\right\|^{\alpha+1}_2\right)^{\frac{1}{\alpha+1}}.
\end{aligned}$$

We can then apply [15, Lemma 5.52] to conclude that

$$S_8 \leqslant Ch^{\frac{1}{2}}\left(\int_0^t \|\boldsymbol{E}\|^{\alpha+1}_{\mathbf{H}^1(\mathbf{curl},\Omega)}\right)^{\frac{1}{\alpha+1}},$$

which is bounded under the previous assumption $\boldsymbol{E} \in L^2((0,T), \mathbf{H}^1(\mathbf{curl}, \Omega))$.

If we collect the estimates of S_i, $i = 1, \ldots, 8$ and apply the Gronwall Lemma, we obtain the final error estimate for the fully discrete approximation scheme.

Theorem 5.1. *Suppose the weak solution of* (5) *and the initial data satisfy*

$$\begin{aligned}
&\boldsymbol{E} \in L^2\left((0,T), \mathbf{H}^1(\mathbf{curl}, \Omega)\right), \qquad &&\boldsymbol{H} \in L^2\left((0,T), \mathbf{H}^1(\Omega)\right), \\
&\boldsymbol{E}_0 \in \mathbf{H}^1(\mathbf{curl}, \Omega), &&\boldsymbol{H}_0 \in \mathbf{H}^1(\Omega).
\end{aligned}$$

Then, the fully discrete finite element approximation satisfies the following error estimates

$$\max_{t\in[0,T]}\left(\left\|\boldsymbol{e}_n^h(t) - \boldsymbol{E}(t)\right\|^2 + \left\|\boldsymbol{h}_n^h(t) - \boldsymbol{H}(t)\right\|^2\right) \leqslant C(\tau + h),$$

where the constant C depends on the initial data $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ and on the weak solution.

Remark 5.1. Similar to [12,14], we needed to assume $\mathbf{H}^1$-regularity of the solutions and of the curl of $\mathbf{E}$ in order to apply the approximation properties of the finite element interpolation operators. From [24,25], we know that $\mathbf{H}^1$-regularity can only be expected a priori for convex polyhedral domains.

6. Numerical experiments

The aim of this section is to present some basic numerical experiments to support the theoretical results. We solve the system (5) numerically using the finite element approximation scheme (27) as described in Section 5. We apply a source function to the right-hand side of Ampère's law in (5), which is chosen such that the exact solution is known. Then we solve the problem numerically for several values of τ and h and compute the error given in Theorem 5.1. The computational domain is $(t, \mathbf{x}) \in [0,1] \times [-1,1]^3$ and the cubic domain is implemented in the mesh generator Gmsh [26].

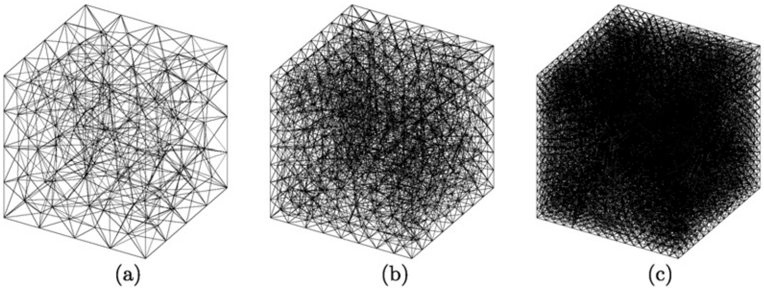

Fig. 1. Tetrahedral meshes on which the problem is solved, constructed in Gmsh [26]. (a) 804 tetrahedra. (b) 5504 tetrahedra. (c) 40 032 tetrahedra.

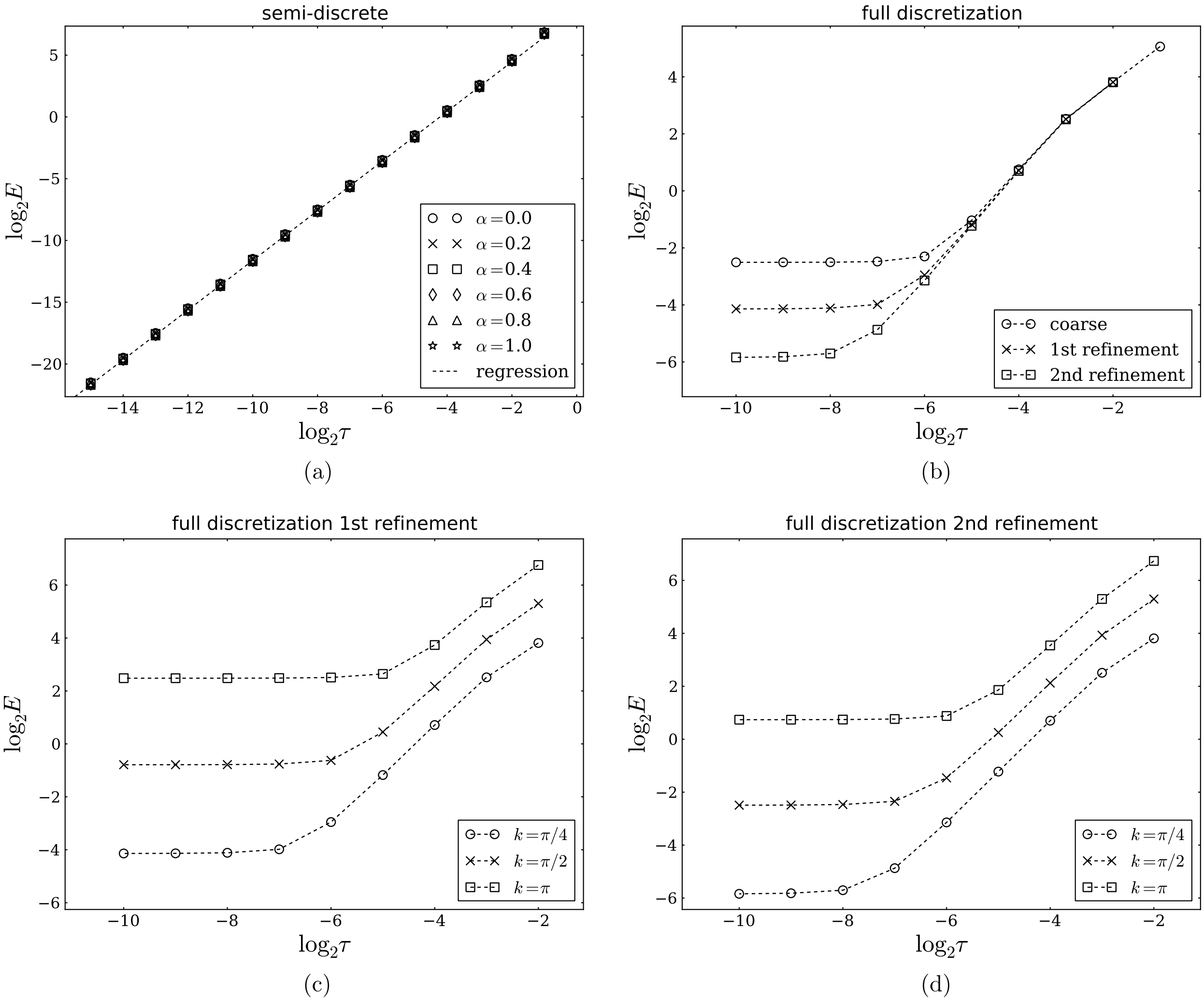

Fig. 2. Results of the numerical experiments. (a) Convergence rate for experiment 1 with α between 0 and 1. The value $\log_2 E$ (E as defined in Eq. (28)) is given as a function of the time step $\log_2 \tau$. The linear regression gives $\log_2 E = \mathbf{2.01} \log_2 \tau + 8.48$, which is better than the theoretically predicted convergence rate. (b) Convergence rate for experiment 2 with $\alpha = 0.6$ and $k = \pi/4$ for the three meshes in Fig. 1. (c) Convergence rate for experiment 2 with $\alpha = 0.6$ for several values of k after one refinement. (d) Convergence rate for experiment 2 with $\alpha = 0.6$ for several values of k after two refinements.

We use the finite element package GetDP [27], where the curl-conforming Nédélec elements described in Section 5 are implemented.

To study the h-dependence of the error, we construct three different meshes by refinement of a coarse mesh. They are depicted in Fig. 1. On every mesh, we solve (27) for $\tau = 2^{-n}$, with $n = 1, \ldots, 15$ and compute the error

$$\boldsymbol{E} = \max_{t\in[0,T]} \left(\left\| \boldsymbol{e}_n^h(t) - \boldsymbol{E}_{\mathrm{ex}}(t) \right\|^2 + \left\| \boldsymbol{h}_n^h(t) - \boldsymbol{H}_{\mathrm{ex}}(t) \right\|^2 \right). \tag{28}$$

The non-linear equation (27) is solved iteratively, using Newton's method.

Experiment 1. As a first experiment, we use as an exact solution

$$\begin{aligned}
\boldsymbol{E}_{\mathrm{ex}}(t,x,y,z) &= (t+1)[z-y, x-z, y-x],\\
\boldsymbol{H}_{\mathrm{ex}}(t,x,y,z) &= -\big(t^2+2t\big)[1,1,1].
\end{aligned}$$

These fields are chosen such that they are elements of the corresponding finite element spaces, i.e., $\boldsymbol{E}_{\mathrm{ex}}$ is of the form $\mathbf{a}+\mathbf{b}\times\mathbf{x}$ on the whole domain. Therefore, the error due to space discretization, which is proportional to $\Pi^h\boldsymbol{E}_{\mathrm{ex}}-\boldsymbol{E}_{\mathrm{ex}}$, vanishes and only the time discretization contributes to the error. We consider them as a mathematical tool to study the validity of the time discretization and no physical interpretation can be given to these fields.

We work with the coarse mesh Fig. 1(a) and vary the time step as $\tau=2^{-n}$, $n=1,\dots,15$. For every value of τ, we compute the error E given by expression (28). The results are given in Fig. 2(a) for several values of α. As expected, we obtain a linear decrease of the error with decreasing time step on logarithmic scale. By fitting a linear regression line to the data we obtain the following convergence rate: $\log_2 E = \mathbf{2.01}\log_2\tau + 8.48$. This corresponds to optimal convergence rate, which is better than the theoretically derived suboptimal convergence in Theorem 5.1. The theoretical result is however more generally valid for domains with less regularity, where the exact solution is also less regular.

Experiment 2. As a second experiment, we apply the following exact solution,

$$\boldsymbol{E}_{\mathrm{ex}}(t,x,y,z) = \big(\sin(2\pi t)+1\big)\begin{bmatrix}\sin(kz)-5\sin(ky)\\ 3\sin(kx)-\cos(kz)\\ 5\cos(ky)-3\cos(kx)\end{bmatrix},$$

$$\boldsymbol{H}_{\mathrm{ex}}(t,x,y,z) = \left(\frac{k}{2\pi}\cos(2\pi t)-kt-\frac{k}{2\pi}\right)\begin{bmatrix}-\sin(kz)-5\sin(ky)\\ -3\sin(kx)+\cos(kz)\\ 5\cos(ky)+3\cos(kx)\end{bmatrix}$$

for $k=\pi/4$, $\tau=2^{-n}$, $n=1,\dots,10$ on the three meshes in Fig. 1. The result for $\alpha=0.6$ is given in Fig. 2(b), where the three meshes are compared. For large values of τ, the error is dominated by the error of time discretization and there is no advantage in refining the mesh. For small values of τ, the error of space discretization dominates. The total error is then independent of the time step and can only be decreased by refining the mesh.

As a final experiment, we compute the error for several values of k, i.e., $k=\pi/4,\pi/2,\pi$ on the first and second refinement. The results are shown in Fig. 2(c) and (d). It is clear that the error is strongly dependent on the spatial variations of the solution. If the oscillations increase, then a finer mesh is needed to approximate the solution.

Acknowledgments

This research was supported by the BOF/GOA-project no. 01G006B7 and by the BOF grant number 01D28807 of Ghent University.

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